Analysis of tangent lines and local extrema.tex

1. Let $f(x) = \ln(x^2 - 8)$
   (a) Find the tangent line to the graph of $f(x)$ at $x_0 = 3$
   (b) Find the $Dom(f(x))$, intervals of increase, and classify the local extrema of $f(x)$.
2. Let $f(x) = \dfrac{x^2 - 4x - 6}{x^2 - 4x}$
   (a) Find the tangent line to the graph of $f(x)$ at $x_0 = -1$
   (b) Find the $Dom(f(x))$, intervals of increase and decrease, and classify the local extrema of $f(x)$.

Solutions

1. (a) To find the tangent line, we need to find the slope and the y-intercept. For the slope, we compute the derivative

$$f'(x) = \frac{2x}{x^2 - 8}$$

$$f'(3) = \frac{6}{9-8} = \frac{6}{1} = 6$$

Once we have the slope, we find the y-intercept knowing that the tangent line takes the same value as the original function at $x_0 = 3$. $f(3) = \ln(9 - 8) = \ln(1) = 0$.

$$0 = 6 * 1 + b$$

$$-6 = b$$

We then have the following tangent line:

$$y = 6x - b$$

(b) To find the domain of the function:

$$x^2 - 8 > 0$$

Then:

$$x^2 > 8$$

$$x < -\sqrt{8} \text{ and } x > \sqrt{8}$$

Therefore, the domain is $(-\infty, -\sqrt{8}) \cup (\sqrt{8}, +\infty)$

To find the intervals of increase and decrease, we differentiate the function:

$$f'(x) = \frac{2x}{x^2 - 8}$$

The derivative takes the value 0 when $x = 0$, but since 0 is outside the domain, it cannot be an extremum. To find the intervals of increase and decrease, we analyze when the derivative is negative or positive. For very large values of x, the derivative is positive and for negative values of x (below $-\sqrt{8}$). For example:

$$f'(-10) = \frac{2 * (-10)}{(-10)^2 - 8} = \frac{-20}{96} < 0$$

$$f'(10) = \frac{2 * (10)}{(10)^2 - 8} = \frac{20}{96} > 0$$

The function decreases: $(-\infty, -\sqrt{8})$ and increases: $(\sqrt{8}, +\infty)$

2.

$$f(x) = \frac{x^2 - 4x - 6}{x^2 - 4x}$$

(a) We find the slope of the tangent line:

$$f'(x) = \frac{(2x-4)(x^2-4x) - (x^2-4x-6)(2x-4)}{(x^2-4x)^2}$$

$$f'(x) = \frac{2x^3 - 8x^2 - 4x^2 + 16x - [2x^3 - 8x^2 - 12x - 4x^2 + 16x + 24]}{(x^2-4x)^2}$$

$$f'(x) = \frac{12x - 24}{(x^2-4x)^2}$$

$$f'(-1) = \frac{12(-1) - 24}{((-1)^2 - 4(-1))^2} = \frac{-36}{(1+4)^2} = -\frac{36}{25}$$

Now, to find the y-intercept:

$$f(-1) = \frac{(-1)^2 - 4(-1) - 6}{(-1)^2 - 4(-1)} = \frac{1 + 4 - 6}{1 + 4} = -\frac{1}{5}$$

Therefore, the tangent line is:

$$-1/5 = -\frac{36}{25}(x) + b$$
$$-1.64 = b$$

The tangent line is:

$$y = -\frac{36}{25}x - 1.64$$

(b) The domain of $f(x)$ are all real numbers except 0 and 4. To find the extremum, we set the derivative equal to 0:

$$\frac{12x - 24}{(x^2 - 4x)^2} = 0$$
$$12x - 24 = 0$$

Therefore, $x = 2$. With this we have 3 points, $0, 2, 4$. We evaluate the derivative between those points:

$$f'(-1) = -36/25 < 0$$
$$f'(1) = -12/25 < 0$$
$$f'(3) = 12/9 > 0$$
$$f'(5) = 36/20 > 0$$

With this information, we can say that the interval of decrease is: $(-\infty, 0) \cup (0, 2)$ and the interval of increase $(2, 4) \cup (4, +\infty)$. Since the function decreases before $x = 2$ and increases after $x = 2$. We say that $x = 2$ is a minimum, but it is local because there are other values for which the function takes a lower value. For example, at $f(-2) = 0.5$, while $f(2) = 2.5$